Chirped Wave Solutions of a Generalized (3+1)-Dimensional Nonlinear Schrödinger Equation

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The exact chirped soliton-like and quasi-periodic wave solutions of the (3+1)-dimensional generalized nonlinear Schrödinger equation including linear and nonlinear gain (loss) with variable coefficients are obtained detailedly in this paper. The form and the behaviour of solutions are strongly affected by the modulation of both the dispersion coefficient and the nonlinearity coefficient. In addition, self-similar soliton-like waves precisely piloted from our obtained solutions by tailoring the dispersion and linear gain (loss).

Key words: (3+1)-D NLSE; Chirp; Ansatz Method; Soliton-Like Wave Solution; Quasi-Periodic Wave Solution.

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1. Introduction

Consider the generalized nonlinear Schrödinger equation (NLSE) in (3+1) dimensions with variable coefficients:

$$\begin{aligned} \mathrm{i}\partial_z\psi = \frac{\beta(z)}{2}\left(\Delta_\perp\psi+\partial_t^2\psi\right)+\gamma(z)|\psi|^2\psi \\ +\mathrm{i}g(z)\psi+\mathrm{i}\chi(z)|\psi|^2\psi, \end{aligned} \tag{1}$$

where $\psi(z,x,y,t)$ is the complex envelope of the electric field, z is the propagation coordinate, $\Delta_\perp=\partial_x^2+\partial_y^2$ represents the transverse Laplacian, and t is the reduced time, i.e., time in the frame of reference moving with the wave packet. The functions $\beta(z)$, $\gamma(z)$, $g(z)$, and $\chi(z)$ are, respectively, the group velocity dispersion (GVD), self-phase modulation (SPM), linear and nonlinear gain (loss). NLSE appears in many branches of physics and applied mathematics [1], such as, for example, in semiconductor electronics [2, 3], optics in nonlinear media [4], photonics [5], plasmas [6], fundament of quantum mechanics [7], dynamics of accelerators [8], mean-field theory of Bose–Einstein condensates [9] or in biomolecule dynamics [10]. During the past several years, many theoretical issues concerning the NLSE have received considerable attention. However, the use of the NLSE is a kind of an idealization of the much more complicate physical problem, therefore other effects as GVD and SPM were discussed in the literature. For example, the using nonlinear optical fibers with inhomogeneous dispersion and nonlinearity for various purposes, including pulse compression [11], stimulation of modulation instability [12], soliton control [13], dispersion management [14], and soliton amplification in long communication lines [15] has been considered theoretically in a number of papers. Recently, great interest has been generated when it was suggested that the (2+1)-dimensional generalized NLSE with varying coefficients may lead to stable 2D solitons [16]. The generalized NLSE (1) in this paper is of considerable importance, as it describes the full spatiotemporal optical solitons, or light bullets, in (3+1) dimensions. When the coefficients are constants, the behaviour of solutions to the NLSE strongly depends on the dimensionality of the problem. In (1+1) dimensions, (1) reduces to

$$\begin{aligned} \mathrm{i}\partial_z\psi = \frac{\beta(z)}{2}\partial_t^2\psi+\gamma(z)|\psi|^2\psi+\mathrm{i}g(z)\psi \\ +\mathrm{i}\chi(z)|\psi|^2\psi. \end{aligned} \tag{2}$$

Equation (2) describes the amplification or absorption of pulses propagating in a monomode optical fiber with distributed dispersion and nonlinearity. In practical applications the model is of primary interest not only for the amplification and compression of optical solitons in inhomogeneous systems, but also for the stable transmission of soliton control. In the limit $\chi(z) \to 0$, i.e. when the nonlinear gain (loss) is comparatively insignificant and so can be neglected, authors of [17, 18] have studied this condition. In this paper, utilizing the ansatz method and a procedure for balancing terms in the expansion, we will find chirped wave solutions of (1).

We define the complex wave ψ of (1) characterized by a nonlinear chirp, resulting from the nonlinear gain [19, 20]:

$$\psi(z,x,y,t) = B(z,x,y,t)\exp\{\mathrm{i}n_0\ln[A(z,x,y,t)] + \mathrm{i}\Phi(z,x,y,t)\}, \tag{3}$$

where n_0 denotes the nonlinear chirp parameter, and A, B, and Φ are real functions of z, x, y, and t. Substituting ψ into (1), we find the following coupled equations:

$$\begin{aligned}&\beta(\partial_y B\partial_y\Phi + \partial_x B\partial_x\Phi + \partial_t B\partial_t\Phi)\\ &+\frac{1}{A}\beta n_0(\partial_y B\partial_y A + \partial_x B\partial_x A + \partial_t B\partial_t A) - \partial_z B + gB\\ &+\chi B^3 - \frac{1}{2A^2}\beta B n_0\big[(\partial_y A)^2 + (\partial_x A)^2 + (\partial_t A)^2\big]\\ &+\frac{1}{2}\beta B\big(\Delta_\perp + \partial_t^2\big)\Phi + \frac{1}{2A}\beta B n_0\big(\Delta_\perp + \partial_t^2\big)A = 0,\end{aligned} \tag{4}$$

$$\begin{aligned}&\gamma B^3 + B\partial_z\Phi - \frac{1}{2A^2}\beta B n_0^2\big[(\partial_y A)^2 + (\partial_x A)^2 + (\partial_t A)^2\big]\\ &+\frac{1}{2}\beta\big(\Delta_\perp + \partial_t^2\big)B + \frac{1}{A}Bn_0\partial_z A\\ &-\frac{1}{2}\beta B\big[(\partial_y\Phi)^2 + (\partial_x\Phi)^2 + (\partial_t\Phi)^2\big]\\ &-\frac{1}{A}\beta B n_0(\partial_y A\partial_y\Phi + \partial_x A\partial_x\Phi + \partial_t A\partial_t\Phi) = 0.\end{aligned} \tag{5}$$

2. Chirped Wave Solutions of the NLSE with First Ansatz

In [19], the author found the chirped bright and dark soliton-like solution for (2). Enlightened by the forms of solutions in this reference, we here seek chirped wave solutions to (1) and assume the functions to be of the first ansatz

$$A = f(z)F(\theta),\ B = \frac{G(z)}{F(\theta)}\frac{dF(\theta)}{d\theta}, \tag{6}$$

$$\begin{aligned}&\theta = k(z)x + l(z)y + m(z)t + \omega(z),\\ &\Phi = a(z)(x^2+y^2+t^2) + b(z)(x+y+t) + e(z),\end{aligned} \tag{7}$$

where f, G, k, l, m, ω, a, b, and e are the parameter functions to be determined later, and $F(\theta)$ is a solution of the following first-order nonlinear ordinary differential equation [21, 22]:

$$\frac{\mathrm{d}F(\theta)}{\mathrm{d}\theta} = \sqrt{\sum_{i=0}^{n} c_i F^i(\theta)}. \tag{8}$$

Then the derivatives with respect to the variable θ become the derivatives with respect to the variable $F(\theta)$ as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\theta} \to \sqrt{\sum_{i=0}^{n} c_i F^i}\,\frac{\mathrm{d}}{\mathrm{d}F},\\ &\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \to \frac{1}{2}\sum_{i=1}^{n} i c_i F^{i-1}\frac{\mathrm{d}}{\mathrm{d}F} + \sum_{i=0}^{n} c_i F^i\frac{\mathrm{d}^2}{\mathrm{d}F^2},\ldots\end{aligned} \tag{9}$$

We remark here that the exact solutions of (1) depend on the explicit solvability of (8); we consider only the case $n = 4$ in this paper:

$$\frac{\mathrm{d}F(\theta)}{\mathrm{d}\theta} = \sqrt{c_0 + c_1F(\theta) + c_2F^2(\theta) + c_3F^3(\theta) + c_4F^4(\theta)}. \tag{10}$$

Substituting (6) and (7) along with (10) into (4) and (5), one obtains a set of conditions on the coefficients and parameters:

$$\begin{aligned}&\frac{\mathrm{d}k}{\mathrm{d}z} = 2ak\beta,\\ &\frac{\mathrm{d}m}{\mathrm{d}z} = 2am\beta,\\ &\frac{\mathrm{d}l}{\mathrm{d}z} = 2al\beta,\\ &\frac{\mathrm{d}a}{\mathrm{d}z} = 2\beta a^2,\\ &\frac{\mathrm{d}b}{\mathrm{d}z} = 2ba\beta,\\ &\frac{\mathrm{d}\omega}{\mathrm{d}z} = b\beta(k+l+m),\\ &\frac{\mathrm{d}e}{\mathrm{d}z} = -\frac{n_0}{f}\frac{\mathrm{d}f}{\mathrm{d}z} - G^2\gamma c_2 + \frac{3}{2}\beta b^2\\ &\qquad + \frac{1}{2}\beta n_0^2 c_2(k^2+l^2+m^2),\end{aligned} \tag{11}$$

$$\begin{aligned}&c_4\big[2\chi G^2 + 3\beta n_0(k^2+l^2+m^2)\big] = 0,\\ &c_0\big[-2\chi G^2 + 3\beta n_0(k^2+l^2+m^2)\big] = 0\end{aligned} \tag{12}$$

with $c_1 = c_3 = 0$. And the linear and nonlinear gain (or loss) must satisfy the following conditions:

$$g = \frac{1}{G}\left(\frac{\mathrm{d}G}{\mathrm{d}z} - \chi c_2 G^3 - 3\beta a G\right), \tag{13}$$

$$\chi = \varepsilon \frac{3n_0\gamma}{2-n_0^2}. \tag{14}$$

Equation (14) implies that the nonlinear chirp parameter n_0 is in fact determined by the ratio $\chi(z)/\gamma(z)$; from the physical point of view, we come to the conclusion that $n_0^2 \neq 2$ for arbitrary nonlinear materials. We consider the most generic case, in which $f(z)$ and $G(z)$ are assumed nonzero and $\beta(z), f(z)$, and $g(z)$ are arbitrary. The following set of exact solutions is found:

$$\begin{aligned} &a = a_0\alpha,\ b = b_0\alpha,\\ &k = k_0\alpha,\ l = l_0\alpha,\ m = m_0\alpha,\\ &\omega = \omega_0 + b_0(k+l+m)\int_0^z \beta \mathrm{d}z, \end{aligned} \tag{15}$$

$$\begin{aligned} G &= G_0 a^{\frac{3}{2}} \exp\left[-\varepsilon\frac{3}{4a}n_0c_2(l^2+m^2+k^2)\right]\\ &\quad\cdot \exp\left(\int_0^z g\mathrm{d}z\right),\\ e &= e_0 + \frac{1}{4a}\left[2c_2(k^2+l^2+m^2)+3b^2\right] - n_0 \ln f,\\ \chi &= \varepsilon\frac{3n_0\gamma}{2-n_0^2} = -\varepsilon\frac{3\beta n_0(l^2+m^2+k^2)}{2G^2}, \end{aligned} \tag{16}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } c_0 = 0,\\ -1 & \text{if } c_4 = 0, \end{cases} \tag{17}$$

and α is the function related only to the GVD coefficient:

$$\alpha = \frac{1}{1-2a_0\int_0^z \beta\mathrm{d}z}. \tag{18}$$

a_0, b_0, l_0, m_0, k_0, ω_0, G_0, and e_0 are free parameters which can be determined by initial or boundary conditions. It should be noted that the function α affects all of the parameters.

The form of solutions depends on what (10) utilized. We note that some solutions of (10), such as the Jacobi elliptic function solutions, can not exist because of the constraint $c_0c_4 = 0$ made in (3) for nonlinear chirp. If we set c_0, c_2, and c_4 in (10) specifically according to [22], we will have several soliton-like and quasi-periodic solutions as follows:

Case I: When $c_0 = 0$, $\varepsilon = 1$, we have

$$\begin{aligned}\Psi_1 &= -G(z)\tanh(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\mathrm{sech}(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{19}$$

with $c_2 = 1$, $c_4 = -1$.

$$\begin{aligned}\Psi_4 &= -G(z)\coth(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\mathrm{csch}(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{20}$$

with $c_2 = 1$, $c_4 = 1$.

$$\begin{aligned}\Psi_2 &= -G(z)\cot(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\csc(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{21}$$

with $c_2 = -1$, $c_4 = 1$.

$$\begin{aligned}\Psi_3 &= G(z)\tan(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\sec(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{22}$$

with $c_2 = -1$, $c_4 = 1$.

Case II: When $c_4 = 0$, $\varepsilon = -1$, we have

$$\begin{aligned}\Psi_7 &= G(z)\tanh(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\cosh(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{23}$$

with $c_0 = -1$, $c_2 = 1$.

$$\begin{aligned}\Psi_8 &= G(z)\coth(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\sinh(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{24}$$

with $c_0 = 1$, $c_2 = 1$.

$$\begin{aligned}\Psi_5 &= G(z)\cot(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\sin(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{25}$$

with $c_0 = 1$, $c_2 = -1$.

$$\begin{aligned}\Psi_6 &= -G(z)\tan(\theta)\\ &\quad\cdot\exp\{\mathrm{i}n_0\ln[f(z)\cos(\theta)] + \mathrm{i}\Phi(z,x,y,t)\}\end{aligned} \tag{26}$$

with $c_0 = 1$, $c_2 = -1$.

From the above results (19)–(26), the functions θ and Φ are written as (7), where G, k, l, m, ω, a, b, and e satisfy (15)–(18), and $\beta(z)$, $f(z)$, and $g(z)$ are arbitrary functions.

We can clearly see from the expressions that when $a(z) = 0$, namely, when the linear chirp effect vanishes, we can obtain readily from (11)–(14) the homogeneous solutions, only the expressions of G, θ, Φ are different:

$$\omega = \omega_0 + b_0(k_0 + l_0 + m_0)\int_0^z \beta \mathrm{d}z,$$
$$G = G_0 \exp\Big[-\varepsilon \frac{3}{2} n_0 c_2 (k_0^2 + m_0^2 + l_0^2)\int_0^z \beta \mathrm{d}z + \int_0^z g\mathrm{d}z\Big], \quad (27)$$
$$e = e_0 + \Big[c_2(l_0^2 + m_0^2 + k_0^2) + \frac{3}{2}b_0^2\Big]\int_0^z \beta\mathrm{d}z - n_0 \ln f,$$
$$\chi = \varepsilon\frac{3n_0\gamma}{2-n_0^2} = -\varepsilon\frac{3\beta n_0(l_0^2+m_0^2+k_0^2)}{2G^2}$$

with (17), and $k = k_0$, $m = m_0$, $l = l_0$, and $b = b_0$ are constants. We can find that the amplitude B is not a constant. This means that the pulse energy is not always conserved.

We can also find from (14) that if the nonlinear chirp parameter $n_0 = 0$, we will have $\chi = 0$. Thus in this case, the main chirp effect contains only linear chirp. Then we have

$$G = G_0 a^{\frac{3}{2}} \exp\Big(\int_0^z g\mathrm{d}z\Big),$$
$$e = e_0 + \frac{1}{4a}\big[2c_2(l^2+m^2+k^2) + 3b^2\big], \quad (28)$$
$$\gamma = -\frac{\beta(k^2+m^2+l^2)}{G^2},$$

and (15) with (18). Therefore, we may think that the nonlinear chirp results from nonlinear gain and this means that we can compensate the nonlinear gain by properly choosing the initial nonlinear chirp in the real optical communication system. We can also see that the change of nonlinear chirp will directly affect the pulse initial phase, pulse amplitude, and the system's linear gain (loss). These characteristics can well be deduced to all solutions. In this case, every solution $F(\theta)$ of (10) is applicable, such as the Jacobi elliptic function solutions.

3. Chirped Wave Solutions to the NLSE with Second Ansatz

Now, let us concentrate on our attention to find the solution of (1) with the second ansatz:

$$A = B = G_1(z)F(\theta) + \frac{G_2(z)}{F(\theta)} \quad (29)$$

with (7), where f, G, k, l, m, ω, a, b, and e are the parameter functions to be determined later, and $F(\theta)$ is a solution of (10). The similar ansatzs (7) and (29) for (1) with the limits $\chi(z) \to 0$ can be found in [18]. Substituting (29) along with (10) into (4) and (5), one also obtains (11) and another set of conditions on the coefficients and parameters:

$$2\chi G_1^2 + 3\beta n_0 c_4(m^2+l^2+k^2) = 0,$$
$$2\chi G_2^2 + 3\beta n_0 c_0(m^2+l^2+k^2) = 0,$$
$$\frac{\mathrm{d}e}{\mathrm{d}z} = -4\gamma G_1 G_2 + \frac{1}{2G_1}\beta(k^2+l^2+m^2) \cdot \big[G_1 c_2(n_0^2-1) - 2G_2 c_4(n_0^2+1)\big] + \frac{3}{2}\beta b^2 - \frac{n_0}{G_1}\frac{\mathrm{d}G_1}{\mathrm{d}z}, \quad (30)$$
$$c_0(2\sqrt{c_4 c_0} - \varepsilon c_2) = 0 \quad (31)$$

with $c_1 = c_3 = 0$ and $\varepsilon = \pm 1$. The linear and nonlinear gain (or loss) must satisfy the following conditions:

$$g = -\beta(k^2+l^2+m^2)n_0 c_2 - 4G_1G_2\chi - 3\beta a + \frac{1}{G_1}\frac{\mathrm{d}G_1}{\mathrm{d}z}, \quad (32)$$
$$\chi = \frac{3n_0\gamma}{2-n_0^2}. \quad (33)$$

We consider the most generic case, in which G_1 is assumed nonzero and $\beta(z)$ and $g(z)$ are arbitrary functions. The following set of exact solutions is found:

$$a = a_0\alpha,\ b = b_0\alpha,$$
$$k = k_0\alpha,\ l = l_0\alpha,\ m = m_0\alpha, \quad (34)$$
$$\omega = \omega_0 + b_0(k+l+m)\int_0^z \beta\mathrm{d}z,$$
$$G_1 = G_0 a^{\frac{3}{2}} \exp\Big[\frac{n_0}{2a}(c_2 - \varepsilon 6\sqrt{c_0c_4})(l^2+m^2+k^2)\Big] \cdot \exp\Big(\int_0^z g\mathrm{d}z\Big),\ G_2 = \varepsilon\sqrt{\frac{c_0}{c_4}}G_1,$$
$$e = e_0 + \frac{1}{4a}\big[(c_2 - \varepsilon 6\sqrt{c_0c_4})(n_0^2-1) \cdot (l^2+m^2+k^2) + 3b^2\big] - n_0 \ln G_1, \quad (35)$$
$$\chi = \frac{3n_0\gamma}{2-n_0^2} = -\frac{3\beta n_0 c_4(m^2+l^2+k^2)}{2G_1^2},$$

where α satisfies (18). Note the relation (31) among the constants c_0, c_2, and c_4. If we set c_0, c_2, and c_4 specifically according to [22], we will have several exact soliton-like and quasi-periodic solutions as follows:

Case I: When $c_0 = 0$, $G_2 = 0$, and G_1 is assumed nonzero, we have

$$\Psi_9 = G_1(z)\mathrm{sech}(\theta) \cdot \exp\{\mathrm{i}n_0 \ln[G_1(z)\mathrm{sech}(\theta)] + \mathrm{i}\Phi(z,x,y,t)\} \tag{36}$$

with $c_2 = 1$, $c_4 = -1$.

$$\Psi_{12} = G_1(z)\mathrm{csch}(\theta) \cdot \exp\{\mathrm{i}n_0 \ln[G_1\mathrm{csch}(\theta)] + \mathrm{i}\Phi(z,x,y,t)\} \tag{37}$$

with $c_2 = 1$, $c_4 = 1$.

$$\Psi_{10} = G_1(z)\csc(\theta) \cdot \exp\{\mathrm{i}n_0 \ln[G_1\csc(\theta)] + \mathrm{i}\Phi(z,x,y,t)\} \tag{38}$$

with $c_2 = -1$, $c_4 = 1$.

$$\Psi_{11} = G_1(z)\sec(\theta) \cdot \exp\{\mathrm{i}n_0 \ln[G_1\sec(\theta)] + \mathrm{i}\Phi(z,x,y,t)\} \tag{39}$$

with $c_2 = -1$, $c_4 = 1$.

Case II: When $c_0 \neq 0$, and G_1 and G_2 are assumed nonzero, we have

$$\Psi_{13} = G_1(z)\left[\tanh(\theta) + \frac{\varepsilon}{\tanh(\theta)}\right] \exp\left\{\mathrm{i}n_0 \ln\left[G_1(z)\left(\tanh(\theta) + \frac{\varepsilon}{\tanh(\theta)}\right)\right] + \mathrm{i}\Phi(z,x,y,t)\right\} \tag{40}$$

with $c_0 = 1$, $c_2 = -2$, $c_4 = 1$.

$$\Psi_{14} = G_1(z)\left[\coth(\theta) + \frac{\varepsilon}{\coth(\theta)}\right] \exp\left\{\mathrm{i}n_0 \ln\left[G_1(z)\left(\coth(\theta) + \frac{\varepsilon}{\coth(\theta)}\right)\right] + \mathrm{i}\Phi(z,x,y,t)\right\} \tag{41}$$

with $c_0 = 1$, $c_2 = -2$, $c_4 = 1$.

$$\Psi_{15} = G_1(z)\left[\tan(\theta) + \frac{\varepsilon}{\tan(\theta)}\right] \exp\left\{\mathrm{i}n_0 \ln\left[G_1(z)\left(\tan(\theta) + \frac{\varepsilon}{\tan(\theta)}\right)\right] + \mathrm{i}\Phi(z,x,y,t)\right\} \tag{42}$$

with $c_0 = 1$, $c_2 = 2$, $c_4 = 1$.

$$\Psi_{16} = G_1(z)\left[\cot(\theta) + \frac{\varepsilon}{\cot(\theta)}\right] \exp\left\{\mathrm{i}n_0 \ln\left[G_1(z)\left(\cot(\theta) + \frac{\varepsilon}{\cot(\theta)}\right)\right] + \mathrm{i}\Phi(z,x,y,t)\right\} \tag{43}$$

with $c_0 = 1$, $c_2 = 2$, $c_4 = 1$.

From the above results (36)–(43), the functions θ and Φ are written as (7), where G_1, k, l, m, ω, a, b, and e satisfy (34)–(35) with (18), and $\beta(z)$ and $g(z)$ are arbitrary functions.

When the linear chirp effect vanishes, i.e. $a(z) = 0$, we can obtain readily from (11) and (30)–(33) the homogeneous solutions, only the expressions of G_1, G_2, θ, and Φ are different:

$$\omega = \omega_0 + b_0(k_0 + l_0 + m_0)\int_0^z \beta \mathrm{d}z,$$

$$G_1 = G_0 \exp\left[n_0\left(l_0^2 + m_0^2 + k_0^2\right)\left(c_2 - \varepsilon 6\sqrt{c_0 c_4}\right) \cdot \int_0^z \beta \mathrm{d}z + \int_0^z g\mathrm{d}z\right],\ G_2 = \varepsilon\sqrt{\frac{c_0}{c_4}}G_1,$$

$$e = e_0 + \frac{1}{2}\left[\left(c_2 - \varepsilon 6\sqrt{c_0 c_4}\right)\left(n_0^2 - 1\right)\left(l_0^2 + m_0^2 + k_0^2\right) + 3b_0^2\right]\int_0^z \beta \mathrm{d}z - n_0 \ln G_1,$$

$$\chi = \frac{3n_0\gamma}{2 - n_0^2} = -\frac{3\beta n_0 c_4\left(m_0^2 + l_0^2 + k_0^2\right)}{2G_1^2} \tag{44}$$

with (31), and $k = k_0$, $m = m_0$, $l = l_0$, and $b = b_0$ are arbitrary constants. If the nonlinear chirp parameter $n_0 = 0$, we have

$$G_1 = G_0 a^{\frac{3}{2}} \exp\left(\int_0^z g\mathrm{d}z\right),\ G_2 = \varepsilon\sqrt{\frac{c_0}{c_4}}G_1,$$
$$e = e_0 + \frac{1}{4a}\left[\left(\varepsilon 6\sqrt{c_0 c_4} - c_2\right)(l^2 + m^2 + k^2) + 3b^2\right],$$
$$\gamma = -\frac{\beta c_4(k^2 + l^2 + m^2)}{G_1^2}, \tag{45}$$

and (34) with (18), the same result is also obtained in [17]. Considering the fact that the constraint (31) vanishes, the other solutions, such as Jacobi elliptic function solutions, come into existence.

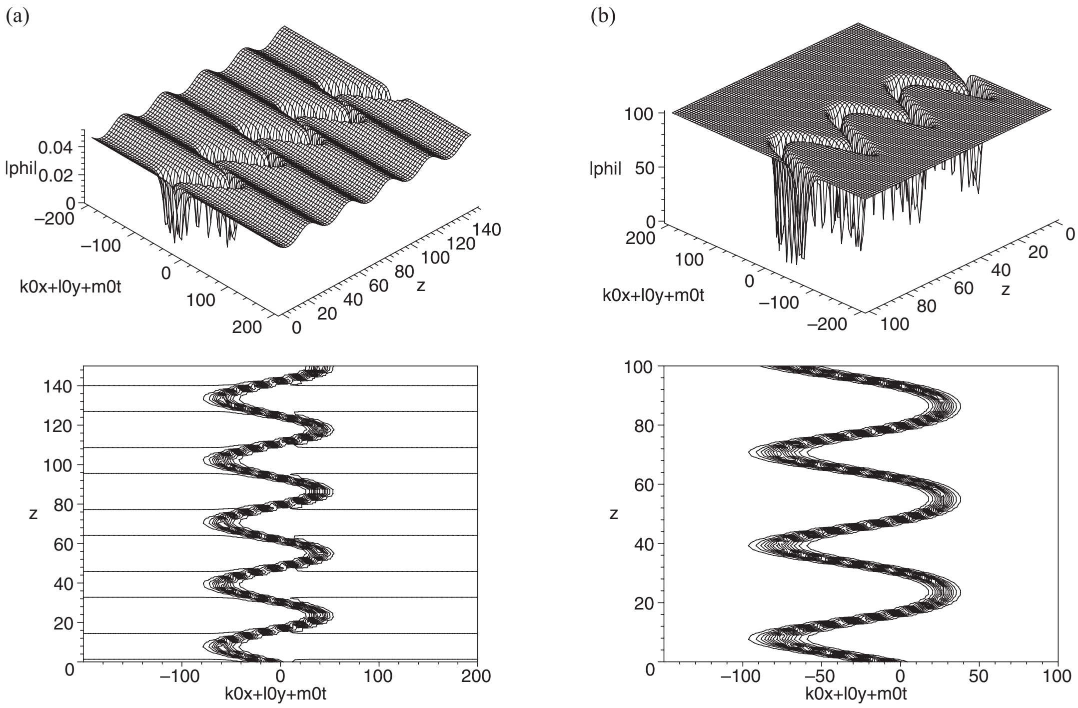

Fig. 1. Distributions of the chirped dark soliton-like solution (19) with and without linear chirp (top row) and a view from above (bottom row): (a) $a_0 = -0.006$, (b) $a_0 = 0$. Other coefficients and parameters: $\beta_0 = 1$, $\eta = 0.2$, $k_0 = -0.1$, $b_0 = 10$, $G_0 = 100$, $n_0 = 1.2$, $\omega_0 = -1$.

4. Self-Similar Wave Solutions

As known to all, self-similar waves are particularly useful in the design of optical fiber amplifiers, optical pulse compressors, and solitary wave base communication links. Remarkably, the self-similar soliton-like waves can be precisely piloted from our obtained solutions by tailoring the GVD and the linear gain (loss). Here, we wish to cite some examples, taking the GVD to be of the form

$$\beta(z) = \beta_0 \cos(\eta z), \tag{46}$$

where β_0 and η are arbitrary constants. In Figure 1, we depict the chirped dark soliton-like solution (19) with the gain

$$g(z) = \frac{3}{2} n_0 \beta_0 (k^2 + l^2 + m^2) \cos(\eta z). \tag{47}$$

In this instance, the corresponding SPM and the nonlinear gain read

$$\chi = \frac{3 n_0 \gamma}{2 - n_0^2} = \frac{3\beta_0 \cos(\eta z) n_0 \left(l_0^2 + m_0^2 + k_0^2\right)\left(2 a_0 \beta_0 \sin(\eta z) - \eta\right)}{2 G_0^2 a_0^3 \eta}. \tag{48}$$

Figure 2 shows the chirped bright soliton-like solution (36) with

$$g(z) = n_0 \beta_0 (k^2 + l^2 + m^2) \cos(\eta z). \tag{49}$$

The corresponding SPM and the nonlinear gain read

$$\chi = \frac{3 n_0 \gamma}{2 - n_0^2} = -\frac{3\beta_0 \cos(\eta z) n_0 \left(l_0^2 + m_0^2 + k_0^2\right)\left(2 a_0 \beta_0 \sin(\eta z) - \eta\right)}{G_0^2 a_0^3 \eta}. \tag{50}$$

Figure 3 presents the chirped quasi-periodic wave solution (39) with

$$g(z) = -n_0 \beta_0 (k^2 + l^2 + m^2) \cos(\eta z). \tag{51}$$

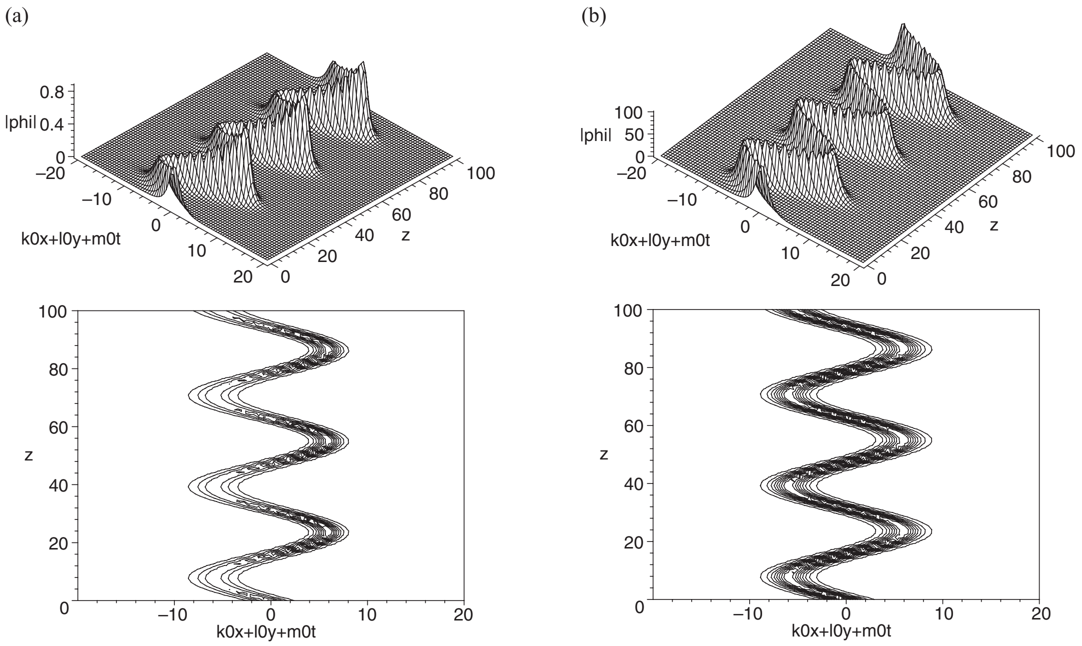

Fig. 2. Distributions of the chirped bright soliton-like solution (36) with and without linear chirp (top row) and a view from above (bottom row): (a) $a_0 = -0.03$, (b) $a_0 = 0$. Other coefficients and parameters: $\beta_0 = 1$, $\eta = 0.2$, $k_0 = -1$, $b_0 = 1.2$, $G_0 = 100$, $n_0 = -1.3$, $\omega_0 = 0$.

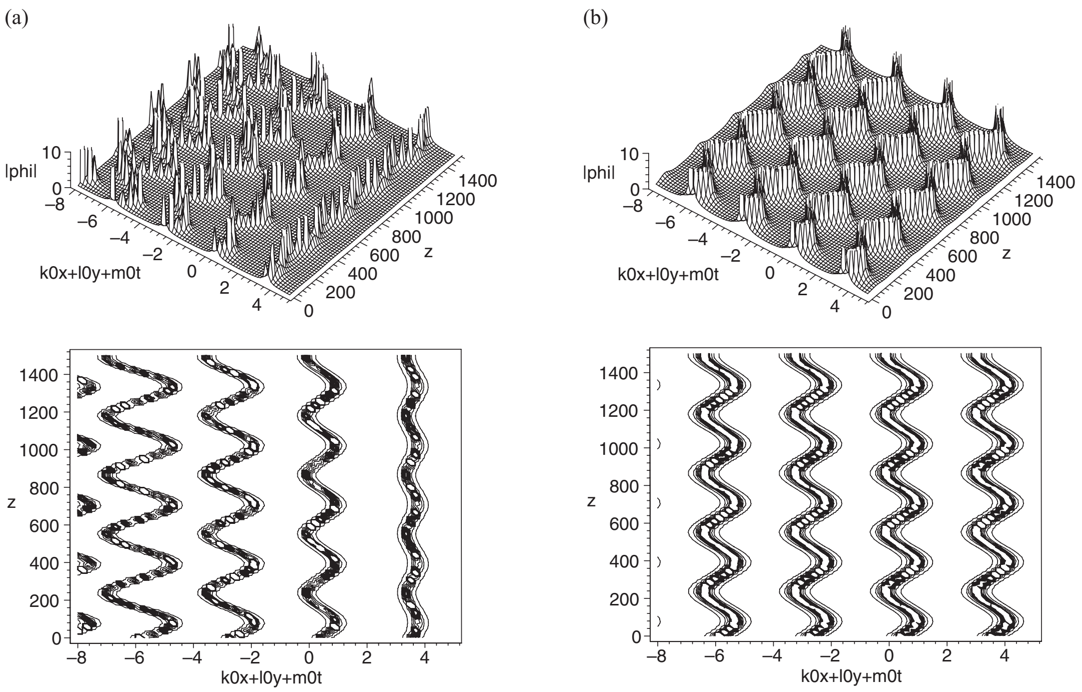

Fig. 3. Distributions of the chirped quasi-periodic wave solution (39) with and without linear chirp (top row) and a view from above (bottom row): (a) $a_0 = 0.001$, $G_0 = 10^4$, (b) $a_0 = 0$, $G_0 = 1$. Other coefficients and parameters: $\beta_0 = 1$, $\eta = 0.02$, $k_0 = 1$, $b_0 = -0.011$, $n_0 = 1$, $\omega_0 = -2$.

The corresponding SPM and the nonlinear gain read

$$\chi = \frac{3n_0\gamma}{2-n_0^2} \tag{52}$$
$$= \frac{3\beta_0\cos(\eta z)n_0\left(l_0^2+m_0^2+k_0^2\right)(2a_0\beta_0\sin(\eta z)-\eta)}{G_0^2a_0^3\eta}.$$

So far, the current situation about the stability of solutions to the generalized (3+1)-dimensional NLSE is somewhat controversial. Some authors consider the stability of radially symmetric structures and do not include the modulation of diffraction [23–25], some authors point out the solutions are not radially symmetric, and the modulation of both the diffraction or dispersion and the nonlinearity is effected concurrently [18]. As can be seen in the insets of Figures 1–3, the features of our results are in agreement with the ones in [18]. The effect of the particular periodic chirp function is to produce a periodic variation along the propagation direction and a monotonic asymmetric change in the transverse directions. The figures also show that the presence of the linear chirp coefficient a_0 significantly changes the nature of solutions. And when the linear chirp vanishes, the pulse shape will be very smooth during the propagation. On the other hand, the nonlinear chirp only makes some decreasing in the pulse intensity. However, experimentally, it might not be easy to maintain such nonlinearly chirped pulses because of the complication of the linear and nonlinear gains in practice.

5. Conclusions

In this paper, by applying the ansatz method and a procedure for balancing terms in the expansion, the chirped soliton-like and quasi-periodic wave solutions of the (3+1)-dimensional generalized nonlinear Schrödinger equation including linear and nonlinear gain (loss) with variable coefficients are obtained detailedly. Unlike the ansatz for the field in [17, 18], here we are concerned with solutions characterized by a nonlinear chirp, see (3), resulting from the nonlinear gain (loss). Remarkably, the self-similar soliton-like waves can be precisely piloted from our obtained solutions by tailoring the GVD and the linear gain (loss). In the presence of linear chirp, the parameters k, l, m, and b are all acquired z dependence. The form and the behaviour of solutions are strongly affected by the modulation of both the GVD and the linear gain (loss) coefficient. The other important feature is that the chirp influences the form of the amplitude. This may provide methods to control the nonlinear gain or absorption by adding initial nonlinear chirp in real systems, and control the change of the pulse amplitude or intensity by adjusting the linear chirp. In conclusion, our analytical results are natural. These findings suggest potential applications in areas such as optical fiber compressors, optical fiber amplifiers, nonlinear optical switches, and optical communications.

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